Introduction to MPRI course 2.33
An introduction

Olivier Bournez
LIX, Ecole Polytechnique, France

MPRI

2021

Menu

Sub-menu

The subject of this course

- THE question
- Motivation 1: Models of Computation
- Motivation 2: Effectivity in Analysis
- Motivation 3: Algebraic Complexity
- Motivation 4: Verification/Control

Consider you preferred function $f : \mathbb{N} \to \mathbb{N}$, or $f : \Sigma^* \to \Sigma^*$.

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Is f computable?

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- verification
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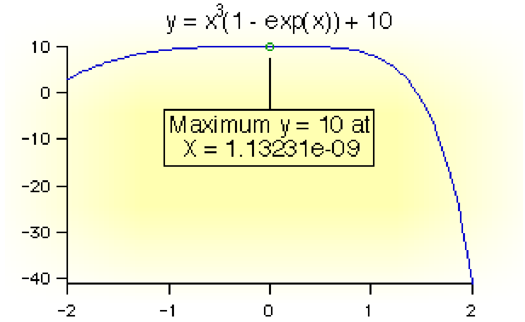
Motivation 1: Models of Computation

NACA Lewis Flight Propulsion Laboratory's Differential Analyser

Question: What is the computational power of this machine?

Sub-menu

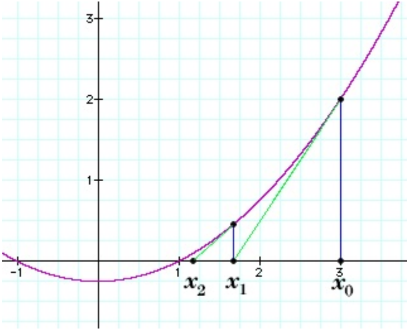
Motivation 2: Effectivity in Analysis



Question: Can we compute the maximum of a continuous function over a compact domain? A point on which it is maximal?

Sub-menu

Motivation 3: Algebraic Complexity



Question: What is the complexity of Newton's method?

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Motivation 4: Verification/Control

- Model $\mathcal{M}$ made of a mixture of continuous/discrete parts.
- Specification ϕ (e.g. reachability property).

Informal question: Can we avoid that?

Formal question:

$$\mathcal{M} \models \phi?$$

Menu

The subject of this course

Back to Foundations of Computer Science

What is a computer ?

Laptop

What is a computer ?

Laptop

Supercomputer

What is a computer ?

Laptop

Supercomputer

Servers

What is a computer ?

Laptop

Supercomputer

Servers

The highest-selling single computer model of all time

source: Guinness World Records

What is a computer ?

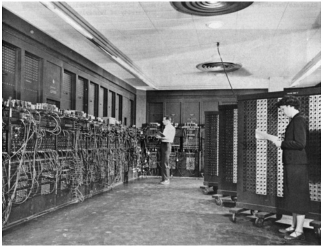
Laptop

Supercomputer

Servers

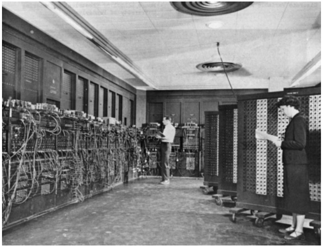
Commodore 64

What is a computer ?



ENIAC

What is a computer ?



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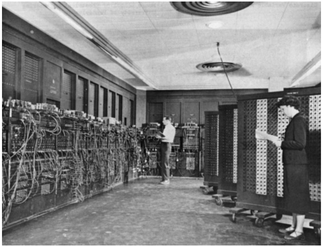
Kelvin's Tide Predicter

What is a computer ?



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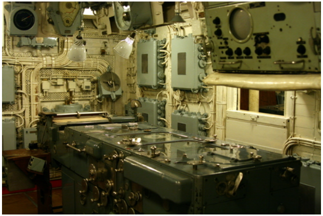
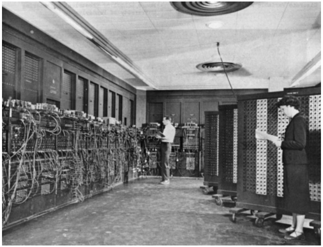


Admiralty Fire Control Table

What is a computer ?



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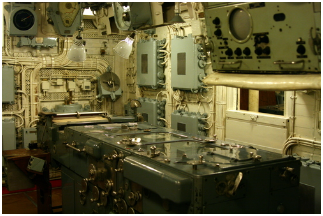
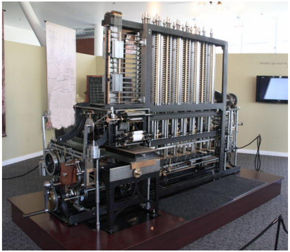


Admiralty Fire Control Table

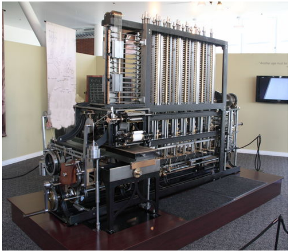
Differential Analyzer

What is a computer ?



Difference Engine

What is a computer ?



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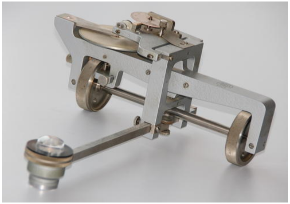
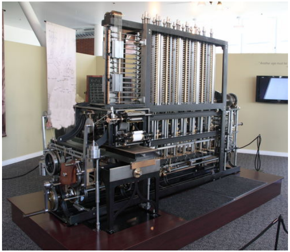


Linear Planimeter

What is a computer ?



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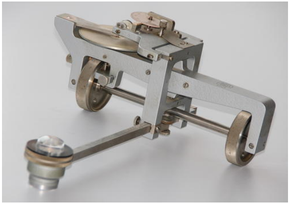

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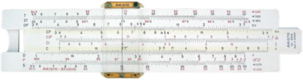

Slide Rule

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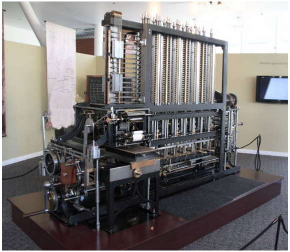

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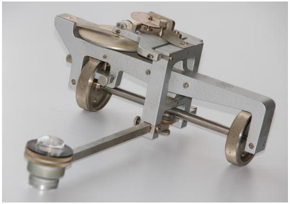

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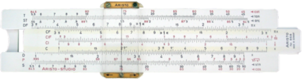
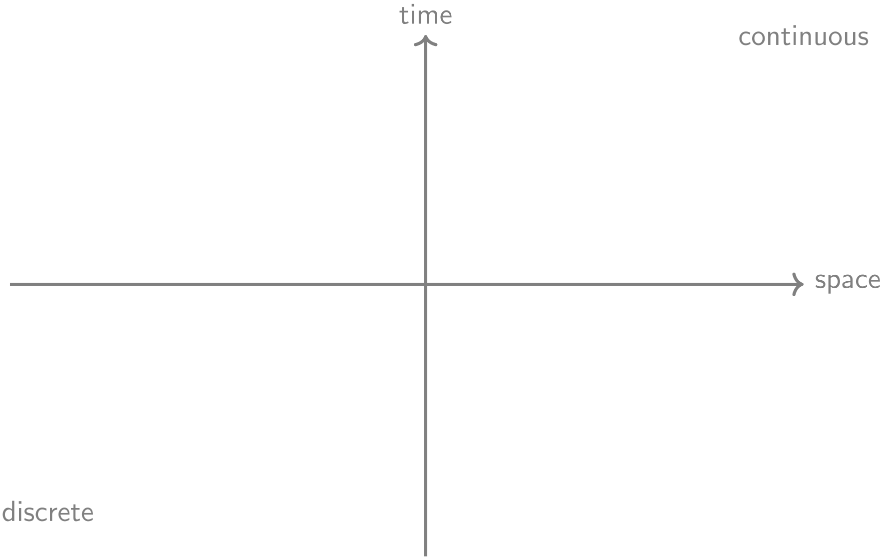


Slide Rule

Antikythera mechanism

A first classification



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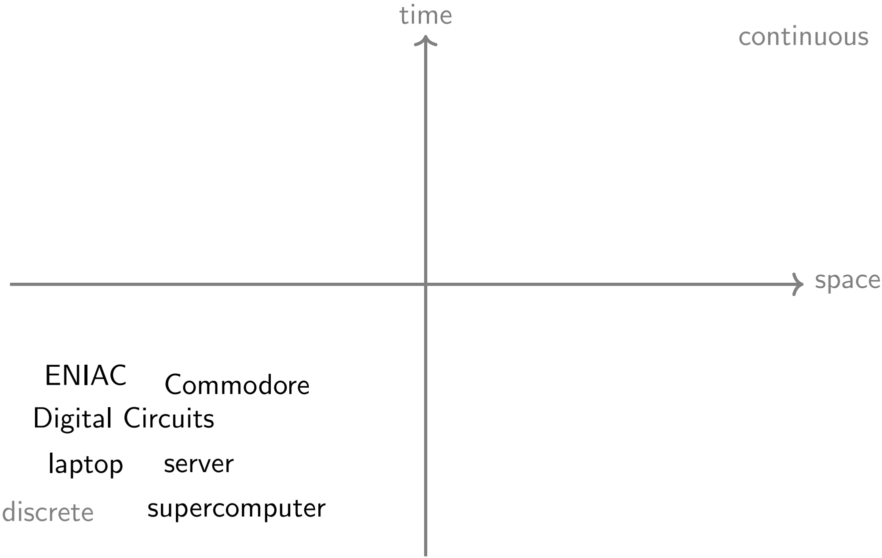

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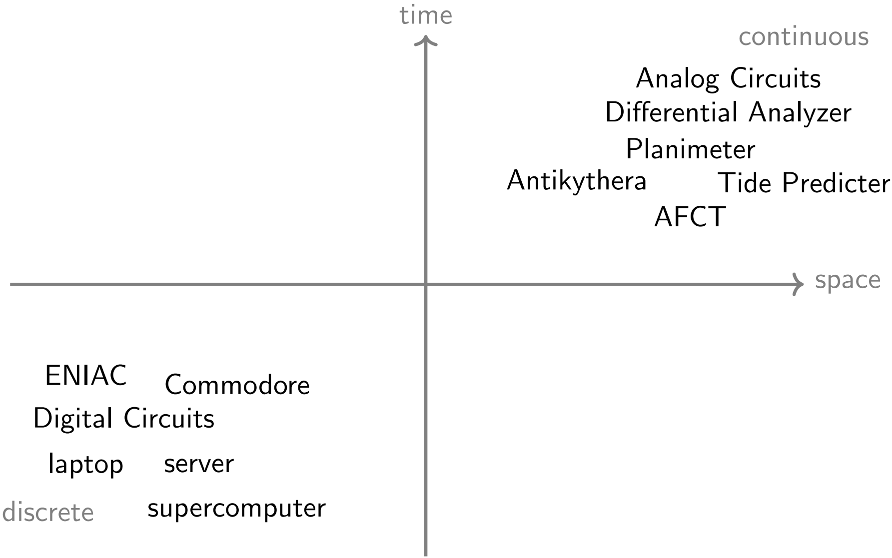

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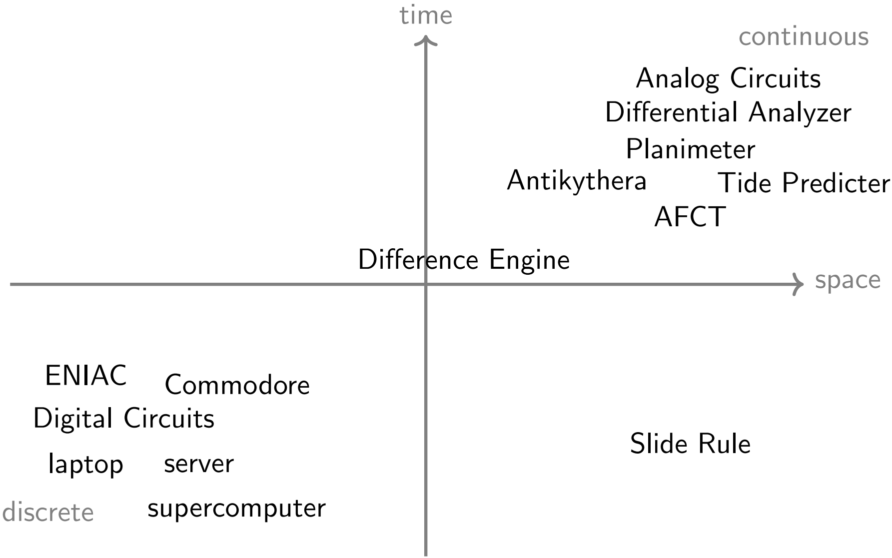

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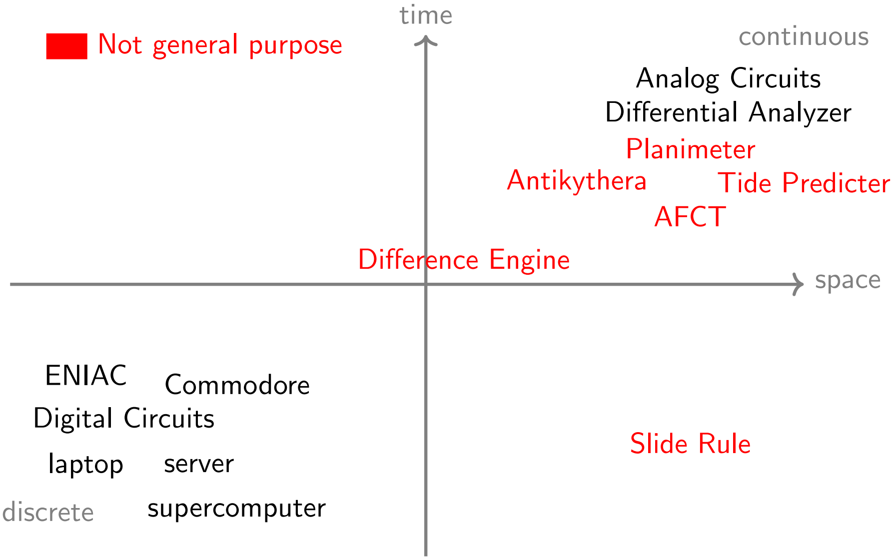

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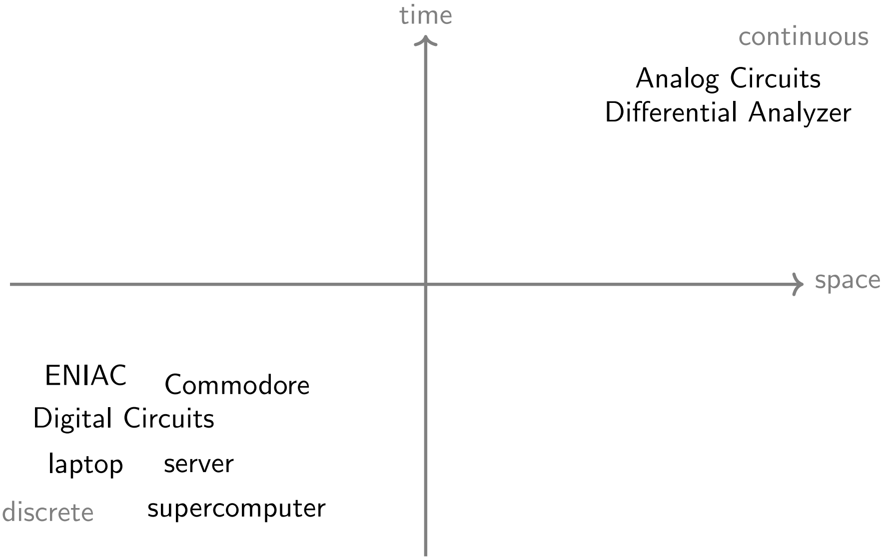

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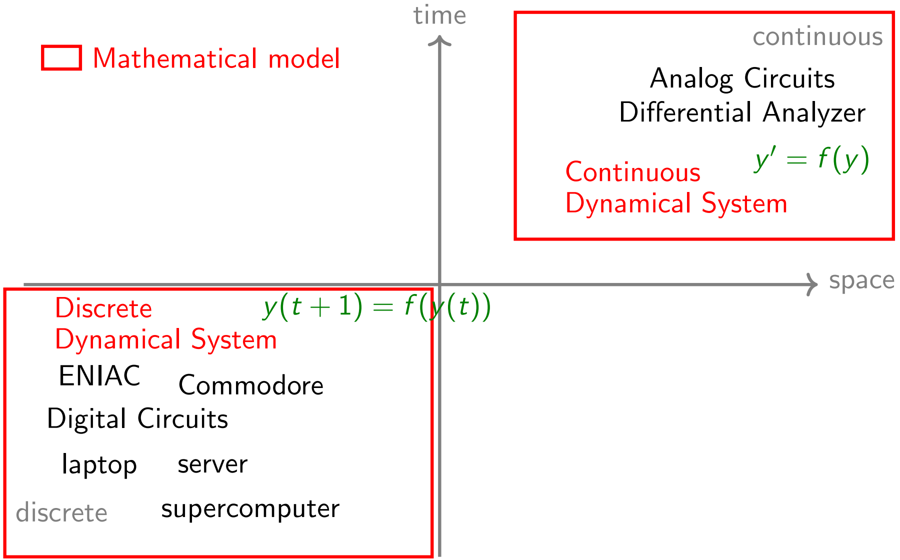

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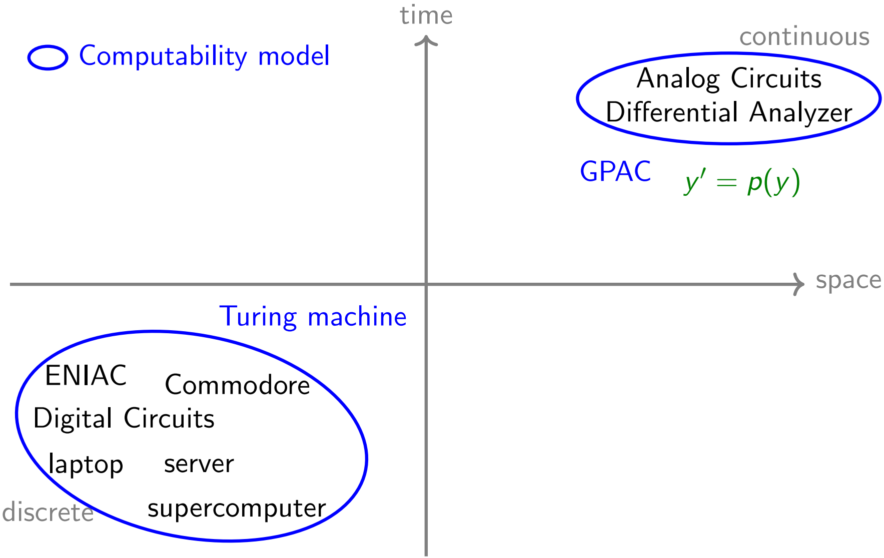

More models!

Physical Computer	**Model**
Laptop, ...	Turing machines
	λ-calculus
	Recursive functions
	Circuits
	Discrete dynamical systems
Differential Analyzer, ...	GPAC
	Continuous dynamical systems

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Church-Turing Thesis

All reasonable models of computation are equivalent.

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Implicit corollary

Some models are too general/unreasonable.

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Shannon's General Purpose Analog Computer

- The **GPAC** is a mathematical abstraction from Claude Shannon (1941) of the **Differential Analyzers**.

- [Graça Costa 03]: This corresponds to **polynomial Ordinary Differential Equations** (pODEs), i.e. continuous time dynamical systems of the form

$$\begin{cases} y(0) = y_0 \\ y'(t) = p(y(t)) \end{cases}$$

  where
  - $y : I \to \mathbb{R}^d, t \in I$
  - and p is a (vector of) polynomials.

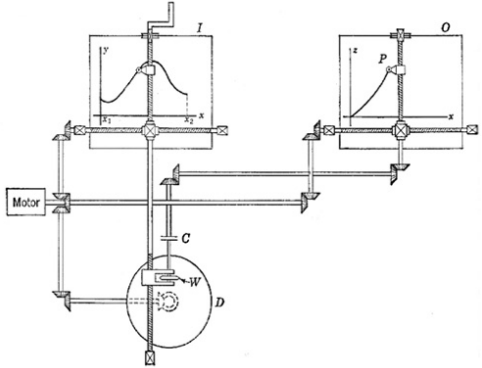
A machine from 20th Century: Differential analyzers

Vannevar Bush's 1938 mechanical Differential Analyser

- Underlying principles: Lord Kelvin 1876.
- First ever built: V. Bush 1931 at MIT.
- Applications: from gunfire control up to aircraft design
- Intensively used during U.S. war effort.
- Electronic versions from late 40s, used until 70s

A machine from 21th Century: Analog Paradigm Model-1

- http://analogparadigm.com
- Fully modular
- Basic version.
  - 4 integrators, 8 constants, 8 adders, 8 multipliers.
  - 14 kgs.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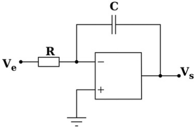
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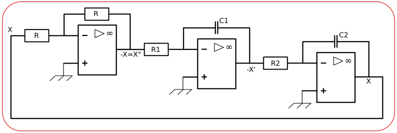
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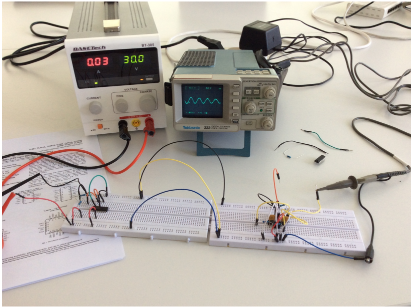
C1
-X'
R2
C2
X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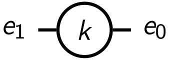
Tektronix

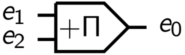
The General Purpose Analog Computer

Shannon's 41 presentation:

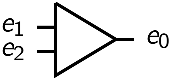
- Basic units:



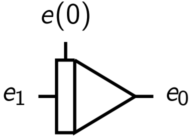
constant: $e_0 = ke_1$



product: $e_0 = e_1 e_2$



summer: $e_0 = -(e_1 + e_2)$



integrator:
$e_0 = -\int_0^t (e_1(u)du + e(0))$

- A function is GPAC-generated if it corresponds to the output of some unit of a GPAC.

The General Purpose Analog Computer

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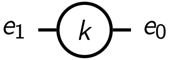

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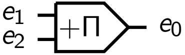

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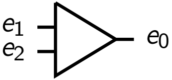

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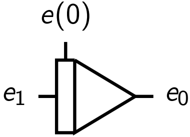

integrator:
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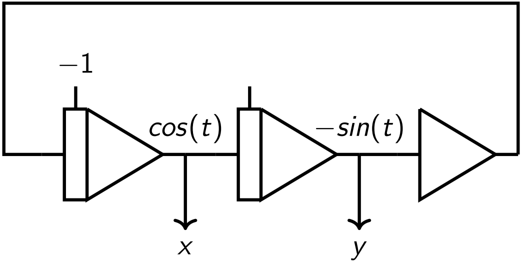
- (Feedback connections are allowed).
- A function is GPAC-generated if it corresponds to the output of some unit of a GPAC.

Cosinus and sinus: $x = \cos(t)$, $y = \sin(t)$



$$\begin{cases} x'(t) = -y(t) \\ y'(t) = x(t) \\ x(0) = 1 \\ y(0) = 0 \end{cases} \Rightarrow \begin{cases} x(t) = \cos(t) \\ y(t) = \sin(t) \end{cases}$$