MPRI COMPLB
Part II
Query complexity.

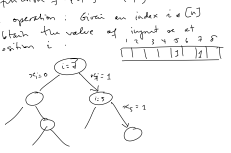
Frame 1

Query Complexity

Decision tree

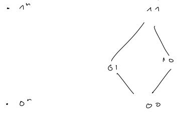
Boolean function $f : \{0,1\}^n \to \{0,1\}$

1 basic operation : Given an index $i \in [n]$ obtain the value of input x at position i.



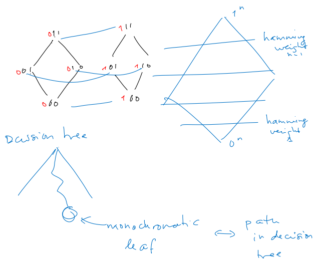
$\boxed{\text{The computation ends}} \to$ every input x leading to this node has $f(x) = b$ for (fixed) $b \in \{0,1\}$

Natural analogue to monochromatic rectangles.



Frame 2

Frame 3



⟷ some variables are fixed $i_1, \dots, i_k$

Def A decision tree T for Boolean function $f : \{0,1\}^n \to \{0,1\}$ is a tree labeled with indices ; it computes f, if the queries made on x determine the value of $f(x)$ correctly

Frame 4

Frame 5

We write $T(x)$ to be the value of the decision tree on x

The $\boxed{\text{cost}}$ of deciding x wit T

$\text{cost}(T; x)$ = depth of the leaf attained by x on T

The cost of T : $\max_{x \in \{0,1\}^n} \text{Cost}(T; x)$

Def Query complexity, or decision tree complexity of f :

$$D(f) = \min_{T:\ \text{decision trees for } f.} \text{cost}(T)$$

Def $R(f)$: randomized decision tree complexity :

$\mathcal{T} = \{T_r\}$ probability p_r of picking tree T_r

Compute using T_r

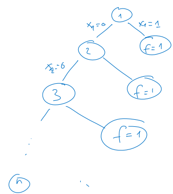
* $\text{Cost}(\mathcal{T}; x) = \max_r \text{cost}(T_r; x)$

* Correctness $\text{Prob}_{r \sim p_r}[T_r(x) = f(x)] \geq \frac{2}{3}$ $\forall x \in \{0,1\}^n$

Frame 6

Example.

$$\text{OR}_n(x) = \begin{cases} 1 & \text{if } \exists i : x_i = 1 \\ 0 & \text{o/w} \end{cases}$$



Easy to see that

$$D(\text{OR}_n) = n$$

$$\text{AND}_n(x) = \begin{cases} 1 & \text{if } x = 1^n \\ 0 & \text{o/w} \end{cases}$$

Frame 7

$$R(f) = \min_{\substack{\mathcal{T}:\ \text{correct} \\ \text{with prob} \\ \frac{2}{3} \text{ on all} \\ \text{inputs}}} \underbrace{\max_{x \in \{0,1\}^n} \text{cost}(\mathcal{T}; x)}_{\text{cost}(\mathcal{T})}$$

Obs
$D(f) \leq n$
$R(f) \leq n$
$R(f) \leq D(f)$

Def $R_0(f)$: zero error randomized D.T.

for this model the cost is in expectation.

$$\overline{\text{cost}}(\mathcal{T}) = \max_x \mathop{\mathbb{E}}_{T \sim \mathcal{T}} \text{cost}(T; x)$$

Prop $\boxed{R(f) \overset{O(1)}{\leq} R_0(f)} \leq D(f)$ $\forall f$

Proof sketch : Let $\mathcal{T}$ be R_0 decision tree with $\mathbb{E}(\text{queries}) = t$.

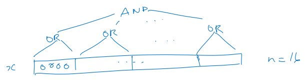
trivial; just use $\mathcal{T} = \{T\}$ where T decides f (deterministically)

Cut off at $3t$. Output 0 at all the cut off branches. → $\text{Prob}[\text{error}] \leq \frac{1}{3}$ Markov's inequality.

Frame 8

TRIBES function :

$\text{AND}_{\sqrt{n}} \circ \text{OR}_{\sqrt{n}}$



AND
OR
OR
OR
x
0000
$\leftarrow \sqrt{n} \rightarrow$
$\leftarrow \sqrt{n} \longrightarrow$
$\sqrt{n}$ blocks
n = 16

Homework (informal)

Give a decision tree for TRIBES.

$$\text{MAJ}_n = \begin{cases} 1 & \text{if } \sum x_i \geq \frac{n}{2} \\ 0 & \text{o/w} \end{cases}$$

$$\text{PARITY}_n = \begin{cases} 1 & \text{if } \sum x_i = 1 \pmod 2 \\ 0 & \text{o/w} \end{cases}$$

Frame 9

Ex OR_n

$D(\text{OR}_n) = n$

$R_0(\text{OR}_n) =?$ n is necessary.

(Key : consider 0^n vs hamming weight 1 inputs.)

$R(\text{OR}_n) < n$?

In fact $R(\text{OR}_n) = \frac{2}{3}n$

$\leq$: look at $\frac{2}{3}$ of the bits (at random)

$\geq$: Ben David's lecture notes Lecture 1 §1.4 page 7

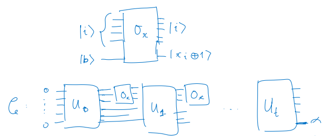
Idea : if we look at $\frac{1}{2}$ the bits then we will make an error with prob $\frac{1}{2}$ if we have seen $\frac{n}{2}$ 0's and the input is one of 0^n or 0^n with 1 bit flipped to 1.

Frame 10

Quantum query complexity

Quantum circuit

Access to the input through query oracle.



Computes a function f with error $\varepsilon = \frac{1}{3}$ if $\forall x$ $\text{Prob}[\text{output bit} = f(x)] \geq \frac{2}{3}$

$\text{Cost}(\mathcal{C}) = t$ # of query operations.

$$Q(f) = \min_{\substack{\mathcal{C}: \\ \varepsilon\text{-correct} \\ \text{for } f}} \text{cost}(\mathcal{C})$$

Thm $Q(\text{OR}_n) = \Theta(\sqrt{n}) \ll R(\text{OR}_n)$

Upper bound : Grover's alg. (1984)

Frame 11

Certificate complexity

Objective: define $C(f)$ that is a lower bound for $D(f)$ i.e. $C(f) \leq D(f)$

Giving a lower bound on D: suffices to compute $C(f)$.

Def A partial assignment $\rho \in \{0,1,*\}^n$

$x \sim \rho$ (x consistent with ρ)

if $\rho_i \neq * \longrightarrow x_i = \rho_i$

Ex $\rho = * * * \textcircled{1} * * *$

$x \sim \rho$: $000\ 1101$

Given a decision tree T and an input x we will associate the partial assignment ρ corresponding to the queries made on input x.

Def ρ is a CERTIFICATE for f if $\exists b \in \{0,1\}$ s.t.

$$f(x) = b \quad \forall x \sim \rho$$

Frame 13

Def For a partial assignment ρ its size $|\rho| = \{i \mid \rho_i \neq *\}$

Def $C(f;x) = \min\{|\rho| : x \sim \rho$ and ρ is a certificate for $f\}$

$$C(f) = \max_x C(f;x)$$

Def $$C_0(f) = \max_{x \in f^{-1}(0)} C(f;x)$$

$$C_1(f) = \max_{x \in f^{-1}(1)} C(f;x)$$

Ex $C_0(OR_n) = n$

$C_1(OR_n) = ?$ 1

Thm $C(f) \leq D(f)$

Frame 12

Example OR_n:

$x = 001001$

$\rho = **1***$

$C(OR_n) = n$

because 0^n requires $\rho = 0 \cdots 0$

Homework

① $C(MAJ_n) = ?$

$$Th_{n,k}(x) = \begin{cases} 1 & \text{if } \sum x_i \geq k \\ 0 & \text{o/w} \end{cases}$$

② $C(Th_{n,k}) = ?$

③ $TRIBES_n = AND_{\sqrt{n}} \circ OR_{\sqrt{n}}$

$C(TRIBES)_n = ?$

Frame 14

Thm $\boxed{D(f) \leq C_0(f)\, C_1(f)} \leq C(f)^2$

Reconvene at 2:30.

~~Proof~~.

Let $\rho_0 \in \mathcal{C}_0$ a set of 0-certificates for f

$\rho_1 \in \mathcal{C}_1$ a set of 1-certificates —"—

$\rho_0 = * * * 1 1 * * * \longrightarrow 0$

$\rho_1 = * * 1 * * * * * \longrightarrow 1$

$00111000 \longrightarrow ?$ impossible.

Prop ρ_0 and ρ_1 MUST overlap (*)

otherwise $\exists y \sim \rho_0$ and $y \sim \rho_1$ impossible since it cannot be that $f(y) = 0$ and $f(y) = 1$

Ex $\rho_0 = * * * 1 1 1 * * *$ } this is ok.

$\rho_1 = * * 0 0 * * * *$

$\rho_1 = * * 0 1 * * * *$ not ok.

"overlap property of certificates."

(*) $\exists i$: $(\rho_0)_i \neq *$

$(\rho_1)_i \neq *$

and $(\rho_0)_i \neq (\rho_1)_i$

Frame 15

Proof Given $\mathcal{C}_0$ a set of 0-certificates for f

$\mathcal{C}_1$ —— 1 ——

Decision tree for f on input x

Pick a certificate $\rho \in \mathcal{C}_0$

Make queries to all the non-$*$ $i_1 \ldots i_k$

If they all agree with ρ:

— output 0

Otherwise:

Restrict $f' = f \restriction_{\text{inputs consistent to the queries so far}}$

If f is constant on this domain output $f(x)$

Recurse on f'

and $\mathcal{C}_0'$ = those 0-certificates from $\mathcal{C}_0$ still consistent with queries

$\mathcal{C}_1'$ = similarly.

Each iteration: make $\leq C_0(f)$ queries.

iterations: $\leq C_1(f)$ since 1-certificates are reducing in size by ≥ 1 at each step.

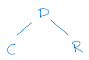
Frame 16

For next time.

What about non-total functions?

- Check that the intersection property does not hold for the following

$$GTH(x) = \begin{cases} 1 & \text{if } \exists i \text{ st. } x_i = 1 \text{ and } i > n/2 \text{ and } w_H(x) = 1 \\ 0 & \text{if } \exists i\ x_i = 1\ i \leq n/2 \text{ and } w_H(x) = 1 \\ \perp & \text{if } w_H(x) \neq 1 \end{cases}$$



Robin Kothari?