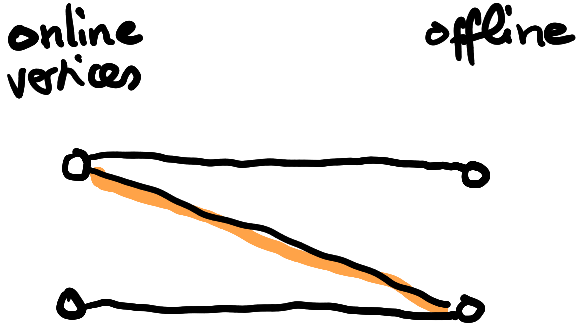
MPRi UNC Oct 19

Online bipartite matching



Greedy deterministic algorithm

match every online vertex to an arbitrary free (unmatched) offline vertex.

it produces a maximal matching.

It is well known that the size of a maximal matching $\geq \frac{1}{2}$ size of maximum matching.

deterministic competitive ratio = $\frac{1}{2}$

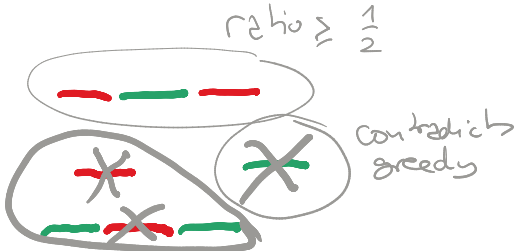
WLOG assume that the final graph admits a perfect matching

Setting online vertices arrive online, together with their adj. edges.

The algorithm can add edges to its current matching. These decisions cannot be undone.

objectif = size of the matching.

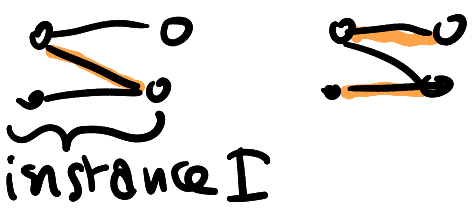




Algorithm Random

match online vertex uniformly at random to a free offline vertex

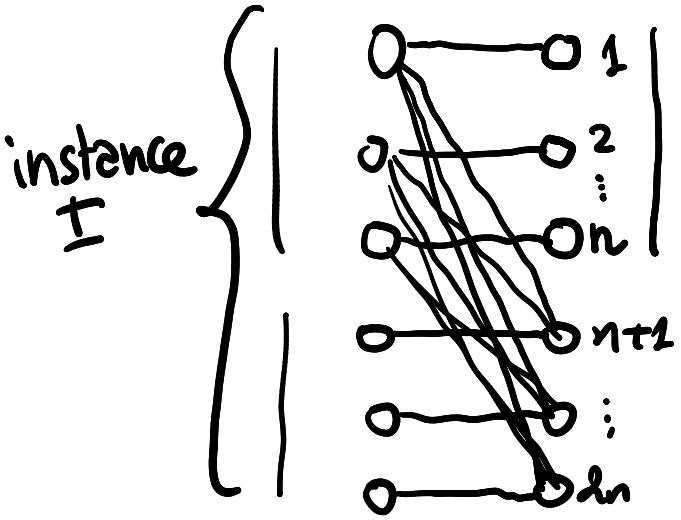
on the previous instance it gives ratio $\frac{3}{4}$ (Good !)



$\mathbb{E}[\text{Random}(I)] = 1.5$

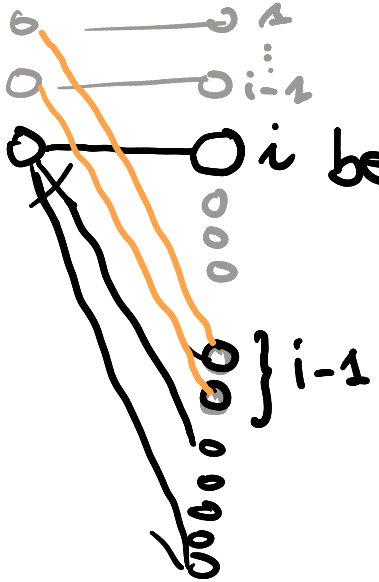
$OPT(I) = 2$

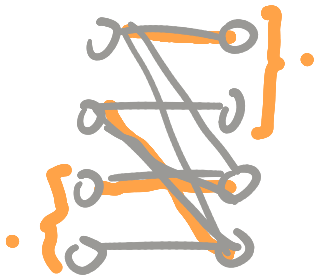
Let's construct an instance which tricks this algorithm



$\forall i \in \{1, \dots, n\}$

$\Pr[\text{offline vertex } i \text{ is matched}] \leq \frac{1}{n+1-(i-1)} = \frac{1}{n-i+2}$





$$\mathbb{E}[\text{Random}(I)] \leq n + \sum_{i=1}^{n} \frac{1}{n+2-i}$$

$$\leq n + \sum_{i=1}^{n} \frac{1}{i} \leq n + \ln(n)$$

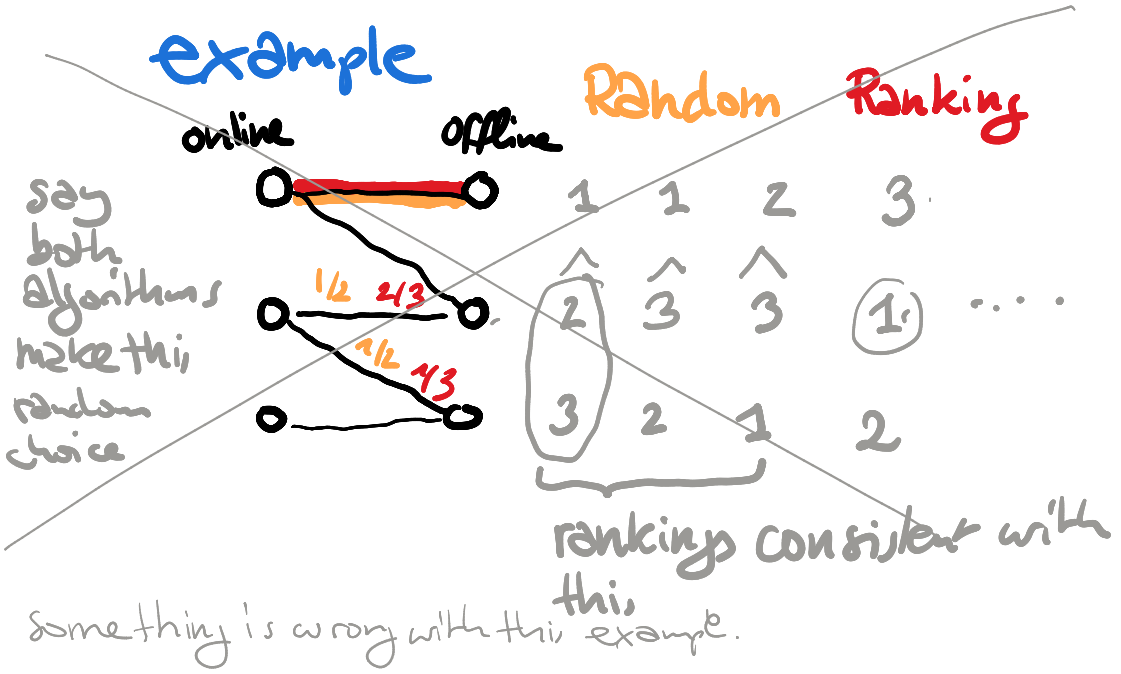
Competitive ratio is arbitrarly close to $\frac{1}{2}$ when n tends to ∞

$OPT(I) = 2n$

ALGORITHM RANKING

choose uniformly at random a ranking on the offline vertices.

Match every online vertex to the free offline vertex of smallest rank.



History

Alg was introduced in 1990 by Karp, Vazirani, Vazirani – proved that Ranking algorithm is $1-\frac{1}{e} \approx 0{,}63$

Flaw in proof was corrected later on, by Birnbaum, Mathieu and others.

Primal Dual analysis of the algorithm 2013 Devanur, Jain, Kleinberg

Define Ranking: $Y_i \in [0,1]$ chosen uniformly at random for every offline vertex i

Define Price: $p(y_i)$

$p: [0,1] \to [0,1]$
p is increasing
$p(1)=1$

p will be chosen later

offline vertices = items i

online vertices = agents. t (time)

has a valuation of 1.

$$\text{Utility}_t := \begin{cases} 0 \text{ if } t \text{ is not matched} \\ 1 - p(Y_i) \end{cases}$$



$$\text{Revenue}_i := p(Y_i)$$

$$\text{size of matching} = \sum \text{Rev}_i + \sum \text{Util}_t$$

$$\text{expect. size of rand. match.} = \sum \mathbb{E}[\text{Rev}_i] + \sum \mathbb{E}[\text{Util}_t]$$

Primal linear program for fractional matching

$$\max \sum_{t,i \in N(t)} x_{t,i}$$

$$\forall t \quad \sum_{i \in N(t)} x_{t,i} \le 1 \quad (\beta_t)$$

$$\forall i \quad \sum_{t : i \in N(t)} x_{t,i} \le 1 \quad (\alpha_i)$$

$$\forall_{t,i \in N(t)} : x_{t,i} \ge 0$$

integral $x_{t,i} = 1$ if t is matched to i

fractional $x_{t,i}$ fraction of i matched to t

dual LP

$$\min \sum_{t,i \in N(t)} \alpha_i + \beta_t$$

$$\forall_{t,i \in N(t)} : \alpha_i + \beta_t \ge 1 \quad (x_{t,i})$$

let's have a break at resume at 10:40

side remark: the primal LP is integral, meaning all vertices of the polytope (= extreme point solutions) are integral
major result in combinatorial optimization.

Thm For the solution produced by the algorithm
$\forall t, i \in N(t)$: $\mathbb{E}[Rev_i] + \mathbb{E}[Util_t] = c := 1 - \frac{1}{e}$

corollary
$\alpha_i = \mathbb{E}[Rev_i]/c$
$\beta_t = \mathbb{E}[Util_t]/c$
give feasible dual solution.
$OPT(I) \cdot \frac{1}{c} \sum(\mathbb{E}[Rev_i] + \mathbb{E}[Util_t]) = \frac{Ranking(I)}{c}$ ⌟

proof
key idea (a) show $\mathbb{E}[Rev_i] + \mathbb{E}[Util_t] \geq c$
on any realization $y_{-i} := (y_{i'})_{i' \neq i}$



key idea (b)
{ Consider two instances,
one with item i (the original instance)
and one without item i

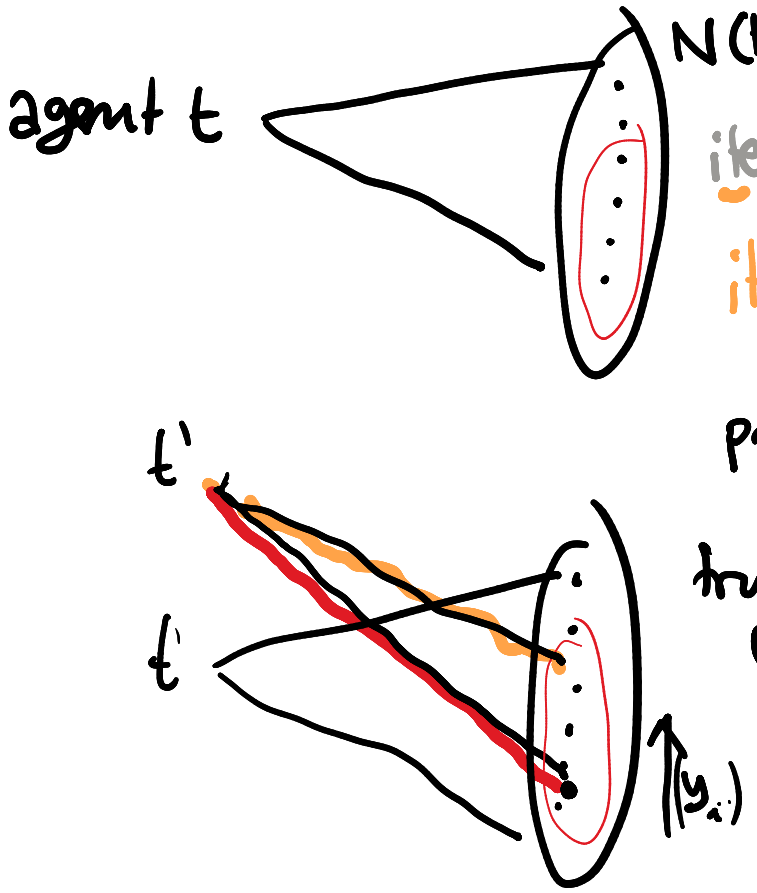
$t \to i$ let $\underline{y}$ be $y_{i'}$ if t is matched to i' in this instance without i
$t \to i'$ and $y = 1$ if t is not matched.

$u = Util_t = 1 - p(y)$ for the instance without i

claim 1: $\mathop{\mathbb{E}}_{y_i}[Rev_i \mid y_{-i}] \geq \int_0^y p(z)\,dz$

claim 2: $Util_t \geq u$ with probability 1
with the instance with i

proof of claim 2



item free in the original instance

item free in the instance without i

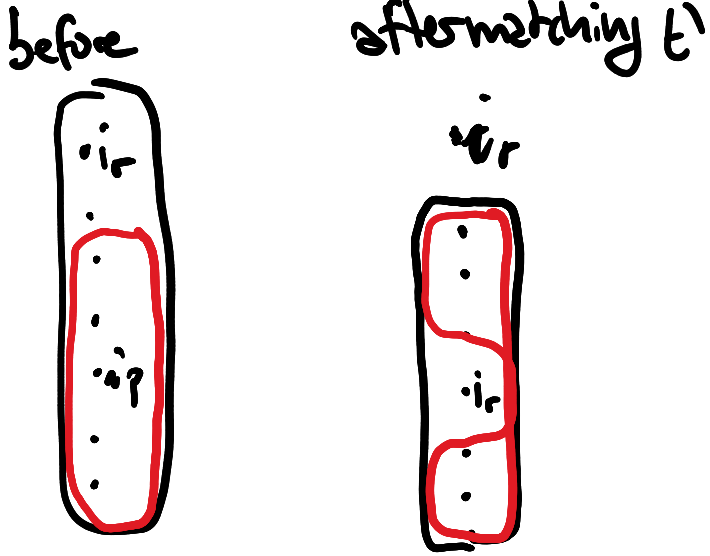
proof by induction on the arrival of the agents

true initially

consider agent t'

suppose in the original instance it is matched to some item i_r and in the instance without i to some item $i_p \neq i_r$

so i_r must have been free in the real instance and not free in the parallel instance



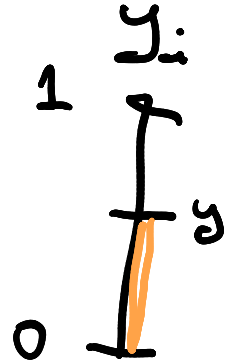
proof claim 1

Condition of $\overline{y_i} < y$
when t arrives in the instance with i

- either i is already matched, $Rev_i = p(y_i)$
- or i is not yet matched, the matchings are identical for both instances

and i is matched to t.

$$\Rightarrow \underset{y_i}{\mathbb{E}}[Rev_i \mid y_{-i}] \geq \int_0^y p(z)\,dz$$



$$\mathbb{E}[Rev_i] + \mathbb{E}[Whilt] \geq 1 - p(y) + \int_0^y p(z)\,dz$$

y depends on y_i

we want $\geq 1 - \frac{1}{e} \quad \forall y \in [0,1]$

in the worst case

$$\forall y \in [0,1] : 1 - p(y) + \int_0^y p(z)\,dz = c := 1 - \frac{1}{e}$$

$$\Rightarrow p'(y) - p(y) = 0$$

$p \propto e^y$

$p(1) = 1$

$$\Rightarrow p(y) = e^{y-1}$$