# Lower Bounds for Models of Computation: Assignment (1/11/2025)

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Answers may be written in English or French. To be submitted in class by November 12th. Use of generative AI is not permitted for this assignment. You may consult any lecture notes or textbooks but please cite your sources if you use them.

## Excercises

1. Let $f$ and $g$ be two total boolean functions, $f : \{0,1\}^n \to \{0,1\}$ and $g : \{0,1\}^m \to \{0,1\}$. Define $F = f \circ g$ as $(f \circ g)(X1, \cdots, X_n) = f(g(X_1), \ldots, g(X_n))$ where $X_i \in \{0,1\}^m$. We say a complexity measure $M$ *composes* if $M(f \circ g) = \Theta(M(f) \cdot M(g))$.
   (a) Show that $C(f \circ g) \leq C(f) \cdot C(g)$.
   (b) Show that $s(f \circ g) = O(s(f) \cdot s(g))$.
   (c) Show that if $f(0^n) = g(0^m) = 0$ then $s(f \circ g, 0^{nm}) \geq s(f, 0^n)s(g, 0^m)$.

2. Let $f : \{0,1\}^n \times \{0,1\}^n$ be a boolean function, and let $g : \{0,1\}^k \times \{0,1\}^k \to \{0,1\}$. Define $F = f \circ g$ as $f \circ g(X, Y) = f(g(X_1, Y_1), \ldots, g(X_n, Y_n))$.
   (a) Give a general upper bound on the randomized communication complexity of $F$ in terms of the query complexity $R(f)$ and the deterministic communication complexity $D^{cc}(g)$. (Give a protocol for $F$ and analyse its complexity.)
   (b) Give a pair of functions $f, g$ as above, where the lower bound on the randomized communication asymptotically matches the upper bound found in the previous exercise. (Don't reprove a lower bound for this exercise. You can cite a result seen in class, or cite a result from one of the textbooks.)
   (c) Give a pair of functions $f, g$ where the lower bound on the randomized communication asymptotically fails to match the upper bound in the previous exercise. (Give a protocol for a composed function whose complexity is asymptotically smaller than the query complexity of the outer function. You can try to compose simple functions such as AND, OR, PARITY...)

3. Let $f$ be a $k$-CNF formula with $m$ clauses. Show that there is a 1-round protocol (i.e., each player sends one message), where Bob sends $\lceil \log_2 m \rceil$ bits and Alice sends $\lceil \log_2 k \rceil$ bits, that computes the result of the Karchmer-Wigderson game for $f$.

   Define the protocol and prove its correctness.

4. Let $X = \{0,1\}^n$ and $Y = \{0,1\}^n$ and let $\mu$ be an arbitrary distribution on $(X, Y)$ such that for all $(x, y)$, $Pr_\mu(x, y) \leq \epsilon$. Let $R$ be arbitrary rectangle $R \subseteq X \times Y$.

   Prove that if $f : X \times Y \to \{0,1\}$ is sampled uniformly at random (i.e., for each $(x, y)$, $f(x, y) = 0$ with probability $1/2$, independently of all other values of $f$), then the Discrepancy of $f$ with respect to $R$ and $\mu$ is more than $\ell \cdot \epsilon \cdot 2^n$ with probability at most $e^{-\Omega(\ell^2)}$.

   *Hint: Use an appropriate version of the Chernoff bound, e.g., the one used in class,* $Pr[\sum_{i=1}^k Z_i - E[\sum_{i=1}^k Z_i] > \delta] \leq e^{-\Omega(\delta^2 k)}$.